AFRL-OSR-VA-TR-2013-0203

Propagation of Uncertainty for Model Validation of Substructured Spacecraft

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University/Company

April 2013
Final Report

DISTRIBUTION A: Approved for public release.

AIR FORCE RESEARCH LABORATORY
AF OFFICE OF SCIENTIFIC RESEARCH (AFOSR)
ARLINGTON, VIRGINIA 22203
AIR FORCE MATERIEL COMMAND

REPORT DOCUMENTATION PAGE			Form Approved OMB No. 0704-0188

1. REPORT DATE (DD-MM-YYYY)	2. REPORT TYPE	3. DATES COVERED (From - To)
25-02-2013	Final Performance Report	Mar 2009 - Nov 2012

4. TITLE AND SUBTITLE
PROPAGATION OF UNCERTAINTY FOR MODEL VALIDATION OF SUBSTRUCTURED SPACECRAFT

5a. CONTRACT NUMBER
FA9550-09-1-0180

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
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5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
University of Wisconsin - Madison
Research and Sponsored Programs
21 N. Park ST STE 6401
Madison, WI 53715-1218

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
AF Office of Scientific Research
875 N. Randolf ST. Room 3112
Arlington, VA 22203

10. SPONSOR/MONITOR'S ACRONYM(S)
AFOSR

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT

13. SUPPLEMENTARY NOTES

14. ABSTRACT
In many situations, it is impossible to perform a system vibration test. If modeling and analysis are to replace system tests, then it is imperative to have confidence in the results. Correlation/validation is the path to providing this confidence and determining the predictability of models used in the decision making process. Within this new validation paradigm, there is no system level test data. Therefore, a probabilistic system correlation must be performed. The product of this research project is a complete and systematic procedure for studying the effects of substructure uncertainty on the test-analysis correlation of complex spacecraft that are validated on a substructure-by-substructure basis, using test and analysis comparisons. The uncertainty is quantified in terms of accepted modal test-analysis correlation metrics, and covariance and relation matrices associated with the differences in the test and FEM frequency response. Linear perturbation analysis is used to relate uncertainty in correlation metrics to uncertainty in substructure matrices. Covariance propagation is then used to propagate substructure uncertainty into the expected correlation metric uncertainty for the system using a Craig-Bampton based component mode synthesis approach.

15. SUBJECT TERMS
Analytical Model Validation; Uncertainty Quantification; Test-Analysis Correlation; Uncertainty Propagation

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	c. THIS PAGE	17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
U	U	U	SAR	37	Daniel C. Kammer
					19b. TELEPHONE NUMBER (Include area code) 608-262-5724

EXECUTIVE SUMMARY

Prior to launch, a spacecraft must have a test-validated finite element model (FEM) that can be used with confidence to predict structural loads, response, performance, etc. The model validation process is comprised of several activities, such as test/analysis correlation using designated metrics, analytical model updating, uncertainty quantification, and determining predictive accuracy. To date, finite element model correlation has usually been done in the modal domain using metrics dictated by agencies such as NASA and the United States Air Force. While an engineer may design a single structure based on drawings, analysis, and experimental results, the reality is that the item produced is just one in an ensemble of structures due to variations and uncertainties in geometry, material parameters, construction, etc. The result is a random population of frequencies and mode shapes. It is common practice to ignore the effects of both model and test uncertainty in model correlation. However, if one does not examine test-analysis correlation relative to uncertainty, very erroneous and dangerous decisions can be made regarding the models ability to make accurate predictions within untested regimes.

In contrast, high-performance precision spacecraft proposed by both the Air Force will require models that are valid to a much higher frequency range for accurate predictions. Within this higher frequency band, spacecraft are modally dense. Due to the corresponding short wavelength vibration patterns, uncertainties have a very large influence on the structural response. Uncertainty then plays an even more important role in modally dense systems at higher frequencies. In addition, due to the high modal density, and the extreme sensitivity of the modes to uncertainty, accepted modal-based methods of model correlation, simply do not work.

As space structures have evolved into larger and more complex systems, ground based vibration tests of the entire spacecraft can become either problematic or impossible. In other situations, such as responsive spacecraft applications, there is no time to perform a full system vibration test. In any case, if a full spacecraft test can be avoided, a great deal of time, effort, and money can be saved. This represents a paradigm shift in space system model validation in which the spacecraft is validated on a substructure-by-substructure basis only. Unavoidable uncertainty in substructure models, connections, and testing will have large, and possibly negative, impact on this new paradigm for model validation. Within this new model validation paradigm, there is no system test data available; therefore strictly speaking, there can be no system model validation, or even correlation. Instead, a probabilistic system correlation must be performed by quantifying uncertainty in the system's substructures and their interfaces, and then propagating it into the system correlation metrics. Several critical questions must be addressed, such as, what constitutes a correlated system model with respect to tolerance for failure, and how does uncertainty and error in the substructures propagate into, and affect the probability of system model correlation?

The overall goal of this project was to develop a new systematic procedure for studying the effects of substructure uncertainty on the system test-analysis correlation of complex spacecraft that are validated on a substructure-by-substructure basis. The uncertainty is quantified in terms of the difference between test and FEM correlation metrics. Due to its universal acceptance within the aerospace community, a Craig-Bampton based component mode synthesis and reduced order modeling approach was utilized.

The first research accomplishment was the development of methods to quantify the uncertainty of a substructure in terms of accepted modal test-analysis correlations metrics. For the low frequency regime, uncertainty in the substructures is quantified in terms of uncertainty in the substructure test-analysis correlation metrics, modal frequency and cross-orthogonality. The substructures are assumed to have been tested in a free-free configuration. Uncertainty in substructure modal mass and stiffness is related to the uncertainty in modal frequency and cross-orthogonality using linear perturbation methods. The uncertainty is defined with respect to the nominal substructure FEM. One of the main contributions of this work is the derivation of the form of the substructure modal mass and stiffness covariance matrices in terms of test- or truth/analysis correlation uncertainty using analytical and numerical experimentation results. Using derived expressions, an analyst can compute covariance matrices for modal mass and stiffness by either specifying the variances of the natural frequencies, the generalized masses, and the off-diagonal terms in each column of the cross-orthogonality matrix, or this data can be based on available test data. Conversely, once substructure is propagated into system level modal matrices, the inverse relations can be used to recover uncertainty in system level correlation metrics. The developed theory was validated using extensive numerical experimentation. A single numerical experiment consisted of Monte Carlo analysis, where at each iteration, the nominal FEM substructure mass and stiffness matrices were randomized using the Maximum Entropy approach. A dispersion level is selected that can be thought of as being analogous to the global fractional uncertainty believed to exist in the matrix, and then the matrix is randomized subject to the constraints of maintaining symmetry and positive definiteness. The advantage of this nonparametric approach, over the usual parameter sensitivity or perturbation methods, is that this randomization process automatically accounts for uncertainties that are not easily described by model parameters, such as model form, geometry, joints, etc. Results showed that off-diagonal cross-orthogonality uncertainty terms are not only zero mean, but normally distributed, and within each column, independent. However, off-diagonal terms are strongly correlated with the corresponding term across the diagonal. Terms on the diagonal follow a generalized chi-square distribution. Based on substructure off-diagonal cross-orthogonality statistics, the $(1-\alpha)$th percentiles can be computed for the diagonal cross-orthogonality terms. This allows the analyst to predict the probability of passing the specified test-analysis correlation criteria.

The second research accomplishment was the development of methods to quantify the uncertainty of a substructure in terms of frequency response. At higher frequencies, spacecraft become modally dense, and the usual modal correlation metrics become useless for uncertainty quantification. Instead, the substructure uncertainty is quantified in terms of covariance and relation matrices associated with the differences in the test/truth and FEM frequency response. At the substructure level, the uncertainty can be obtained from a series of vibration tests and associated test-analysis correlations, a database of previous test-analysis correlation results of similar structures, specified by the user, etc. Linear perturbation analysis and modal filters were used to derive expressions relating uncertainty in frequency response to uncertainty in modal impedance, relative to the nominal model modal basis. Once the substructure uncertainty is propagated into the system level modal impedance, these same expressions can be inverted to recover the system level frequency response covariance and relation matrices. These two matrices completely specify the second order statistical properties of the frequency response. Methods were then developed to recover uncertainty in system level frequency response magnitude and phase. The developed theory was validated using extensive numerical experimentation.

The third research accomplishment was the development of a new systematic approach for propagating substructure uncertainty into uncertainty in system level modal matrices. The objective is to be able to quantify the level of test-analysis correlation required at the substructure level to produce acceptable correlation for the system. Once substructure modal matrix uncertainties are recovered using previously described metrics, linear covariance propagation is used to propagate uncertainty in substructures into the expected uncertainty in correlation for the system. Due to its universal acceptance within the aerospace community, a Craig-Bampton based component mode synthesis and reduced order modeling approach was utilized. It is standard in many finite element codes used by the aerospace community, and it is useful in this application because it neatly separates a substructure interface from its interior. Uncertainties in substructure modal matrices are first propagated into substructure Craig-Bampton representation matrices. In order to propagate uncertainty, the matrix equations must be put in the proper form using a vectorization approach, in which the columns of a matrix are stacked column-wise, combined with the use of Kronecker products, and oblique modal projectors. The projectors spatially filter out any undesired modal contributions. Memory requirements and computational effort can be minimized by taking advantage of the fact that both substructure and system matrices are symmetric. Therefore, only lower triangular terms in uncertainty matrices must be propagated. Redundant terms can be either removed or recovered using elimination and duplication matrices. In the case of frequency response, quantifying uncertainty in terms of Craig-Bampton substructure matrices, and then propagating into the system using component mode synthesis, has a significant advantage over methods that use direct assembly of substructure frequency response, in that translations and rotations at substructure interfaces do not have to be measured. Low and high modal density examples are investigated, and the results are substantiated using Monte Carlo analysis.

The product of this project is a complete and systematic procedure for studying the effects of substructure uncertainty on the test-analysis correlation of complex spacecraft that are validated on a substructure basis. In many situations, it is either impossible to perform a system vibration test, or it is highly desirable to avoid one to conserve time and resources. Therefore, a probabilistic system correlation must be performed by quantifying uncertainty in the system's substructures, and then propagating it into the system correlation metrics. This work is significant to the Air Force because it must make critical decisions concerning space structure performance and survivability based on the results of test-analysis correlation.

The work performed during this project was conducted by:

Dr. Daniel C. Kammer, Professor
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Project research resulted in four publications:

Kammer, D. C., and Nimityongskul, S., "Propagation of Uncertainty in Test-Analysis Correlation of Substructured Spacecraft," *Journal of Sound and Vibration*, Vol. 330, No. 6, 1211-1224, 2010.

Kammer, D. C., and Krattiger, D., "Propagation of Spacecraft Free-Interface Substructure Uncertainty into System Test-Analysis Correlation," *Journal of Vibration and Acoustics*, Vol. 134, No. 2, Oct. 2012, 051014.

Kammer, D. C., and Krattiger, D., "Propagation of Uncertainty in Substructured Spacecraft using Frequency Response," accepted *AIAA Journal*, Jul. 2012.

Nimityongskul, A.P., Kammer, D.C., Lacy, S., and Babuska, V., "Frequency Domain Test-Analysis Correlation in the Presence of Model Uncertainty," Structural Dynamics, Vol. 3, Conference Proceedings of the Society of Experimental Mechanics Series, Vol. 12, pp. 403-418, 2011.

Project research resulted in one doctoral research thesis:

Nimityongskul, Aaron P "A Frequency Domain Approach to Pretest Analysis and Model Updating in the Presence of High Modal Density," ProQuest Dissertations And Theses; Thesis (Ph.D.)--The University of Wisconsin - Madison, 2010.; Publication Number: AAT 3436998; ISBN: 9781124363417; Source: Dissertation Abstracts International, Volume: 72-01, Section: B, 201 p.

PROPAGATION OF UNCERTAINTY FOR MODEL VALIDATION OF SUBSTRUCTURED SPACECRAFT

1.0 INTRODUCTION

Prior to launch, a spacecraft must have a test-validated finite element model (FEM) that can be used with confidence to predict structural loads, response, performance, etc. The model validation process is comprised of several activities, such as test/analysis correlation using designated metrics, analytical model updating, uncertainty quantification, and determining predictive accuracy. To date, finite element model correlation has usually been done in the modal domain. Test and analysis modal frequencies are compared directly, while mode shapes are compared using orthogonality and cross-orthogonality computations. The use of these metrics, and the required values for acceptable correlation, are dictated by agencies such as NASA and the United States Air Force. The requirements differ, depending on the agency. The Air Force, for example, requires test-analysis frequency errors less than or equal to 3.0%, cross-generalized mass values greater than 0.95, and coupling terms between modes of less than 0.10 in both cross-orthogonality and orthogonality.

Recently, quantification of model uncertainty and its propagation through large numerical simulations has been the focus of investigation. While an engineer may design a single structure based on drawings, analysis, and experimental results, the reality is that the item produced is just one in an ensemble of structures due to variations and uncertainties in geometry, material parameters, construction, etc. The result is a random population of frequencies and mode shapes. Besides model uncertainty, there is a corresponding uncertainty and error in the measured test data. In the low frequency regime of modal-based test-analysis correlation and model updating, it is common practice to ignore the effects of both model and test uncertainty. However, if one does not examine test-analysis correlation relative to uncertainty, very erroneous and dangerous decisions can be made regarding the models ability to make accurate predictions within untested regimes.

In contrast with most current applications, high-performance, precision space vehicles proposed by both the Air Force and NASA will need extremely low-level on-orbit vibration environments. Precision spacecraft require models that are valid to a much higher frequency range for accurate predictions. Within this higher frequency band, the spacecraft is modally dense. Due to the corresponding short wavelength vibration patterns, uncertainties have a very large influence on the structural response. Uncertainty then plays an even more important role in modally dense systems at higher frequencies. In addition, due to the high modal density, and the extreme sensitivity of the modes to uncertainty, the accepted modal-based methods of model correlation, such as cross-orthogonality, modal frequency comparisons, etc., simply do not work because modes cannot be differentiated. Therefore, in this investigation, model uncertainty is quantified in terms of test-analysis frequency response comparisons.

As space structures have evolved into larger and more complex systems, ground based vibration tests of the entire spacecraft can become either problematic or impossible. In other situations, such as responsive spacecraft applications, there is no time to perform a full system vibration test. In any case, if a full spacecraft test can be avoided, a great deal of time, effort, and money can be saved. This represents a paradigm shift in space system model validation in which the spacecraft is validated on a substructure-by-substructure basis only. Unavoidable uncertainty in substructure models, connections, and testing will have large, and possibly negative, impact on this new paradigm for model validation. The research community has started to investigate the effects of substructure uncertainty on synthesized system response using component mode synthesis techniques (CMS). The CMS approach has been used for years to solve large structural dynamics problems, and is built into many standard finite element analysis codes. None of the work mentioned in this area addresses the test-analysis correlation component of model validation, at either the substructure or system level.

Within this new model validation paradigm, there is no system test data available; therefore strictly speaking, there can be no system model validation, or even correlation. Instead, a probabilistic system correlation must be performed by quantifying uncertainty in the system's substructures and their interfaces, and then propagating it into the system correlation metrics. Several critical questions must be addressed, such as, what constitutes a correlated system model with respect to tolerance for failure, and how does uncertainty and error in the substructures propagate into, and affect the probability of system model correlation? In this work, a new systematic procedure is developed for studying the effects of substructure uncertainty on the system test-analysis correlation of complex spacecraft that are validated on a substructure-by-substructure basis. The uncertainty is quantified either in terms of the difference between test and FEM frequency response, or accepted modal correlation metrics. At the substructure level, the uncertainty can be obtained from a series of vibration tests and associated test-analysis correlations, a database of previous test-analysis correlation results of similar structures, specified by the user, etc.

The objective is to be able to quantify the level of test-analysis correlation required at the substructure level to produce acceptable correlation for the system. Due to its universal acceptance within the aerospace community, a Craig-Bampton (CB) based component mode synthesis and reduced order modeling approach is utilized. The proposed procedure is of special interest in systems with high modal density, where modal methods do not work. Understanding substructure correlation requirements will positively impact the speed of the loads analysis process.

The product of this project is a complete and systematic procedure for studying the effects of substructure uncertainty on the test-analysis correlation of complex spacecraft that are validated on a substructure basis. In many situations, it is either impossible to perform a system vibration test, or it is highly desirable to avoid one to conserve time and resources. Therefore, a probabilistic system correlation must be performed by quantifying uncertainty in the system's substructures, and then propagating it into the system correlation metrics. This work is significant to the Air Force because it must make critical decisions concerning space structure performance and survivability based on the results of test-analysis correlation.

2.0 SIGNIFICANT RESEARCH ACCOMPLISHMENTS

2.1 Quantification of Uncertainty Using Modal Correlation Metrics

Uncertainty in the substructures is quantified in terms of uncertainty in the substructure test-analysis correlation metrics, modal frequency and cross-orthogonality. In this case, the substructures are assumed to have been tested in a free-free configuration. The uncertainty is defined with respect to the nominal substructure FEM. The nominal substructure modes are assumed to be normalized with respect to mass, such that the nominal substructure modal mass and stiffness are $m = I$ and $k = \Omega$, where Ω is the diagonal matrix of nominal eigenvalues. The "truth" model of the substructure can also be represented in nominal modal coordinates as m_T and k_T. Using linear perturbation theory, uncertainty in the correlation metrics, cross-orthogonality $\Delta\gamma$, and frequency $\Delta\Omega$, can be related to uncertainty in the substructure modal mass and stiffness Δm and Δk.

One of the main contributions is the derivation of the form of the substructure modal mass and stiffness covariance matrices in terms of test- or truth-analysis correlation uncertainty using analytical and numerical experimentation results. It was assumed that the expected values of the uncertainty in the substructure physical mass and stiffness matrices, $E(\Delta M)$ and $E(\Delta K)$, are both zero. The same is then true for the modal matrices, $E(\Delta m) = 0$ and $E(\Delta k) = 0$. This then implies that the expected value of the uncertainty in the generalized masses, $E(\Delta_{Mjj}) = 0$, and the expected value of uncertainties in natural frequencies, $E(\Delta\omega_j) = 0$. Uncertainties in mass and stiffness were assumed to be independent, and therefore, uncorrelated. It can then be shown that the variance of the off-diagonal substructure modal mass uncertainty terms is given by

$$E\left(\Delta m_{ij}^2\right) = \frac{\Omega_j - \Omega_i}{\Omega_j + \Omega_i}\left[E\left(\Delta\gamma_{ij}^2\right) - E\left(\Delta\gamma_{ji}^2\right)\right] \tag{1}$$

The variance of the diagonal modal stiffness uncertainty can be computed using

$$E\left(\Delta k_{jj}^2\right) = 4\Omega_j E\left(\Delta\omega_j^2\right) - \Omega_j^2 E\left(\Delta_{Mjj}^2\right) \tag{2}$$

while the variance of off-diagonal terms Δk_{ij} is given by

$$E\left(\Delta k_{ij}^2\right) = \frac{\Omega_j - \Omega_i}{\Omega_j + \Omega_i}\left[\Omega_j^2 E\left(\Delta\gamma_{ji}^2\right) - \Omega_i^2 E\left(\Delta\gamma_{ij}^2\right)\right] \tag{3}$$

Using numerical experiments, it was shown that the off-diagonal cross-orthogonality uncertainty terms $\Delta\gamma_{ij}$ are not only zero mean, but normally distributed, and within each column, independent. It was also demonstrated that the term $c_j = 1 - \gamma_{jj}^2$ is the sum of the squares of $n_m - 1$ zero mean, normally distributed variables $\Delta\gamma_{ij}$. If the terms $\Delta\gamma_{ij}$ all had unit variance, c_j would be represented by a chi-square distribution with $n_m - 1$ degrees of freedom. In this case, however, the off-diagonal terms $\Delta\gamma_{ij}$ will have different non-unit variances. Therefore, c_j follows a generalized chi-square distribution. In general, the number of degrees of freedom is not equal to $n_m - 1$ because many of the nominal FEM modes do not couple strongly with the *j*th truth mode, meaning many of the terms $\Delta\gamma_{ij}$ are small. As the number of terms that significantly contribute increases, the probability distribution approaches a normal distribution. In terms of standard normal variables z_i, c_j can be written as

$$c_j = 1 - \gamma_{jj}^2 = \sum_{\substack{i=1 \\ i \neq j}}^{n_m} \sigma_i^2 z_i^2 \tag{4}$$

where $\sigma_i^2 = E\left(\Delta\gamma_{ij}^2\right)$ are the off-diagonal cross-orthogonality variances recovered for the jth column. A distribution for c_j can then be constructed by taking a linear combination of single degree of freedom chi-square distributions $\chi^2(1)$. The $(1-\alpha)$th percentile for c_j can easily be computed and then the corresponding value for the jth cross-generalized mass, given by $\gamma_{jj\alpha} = \sqrt{1 - c_{j\alpha}}$, can be compared to designated correlation metric criteria.

A simple representation of a communications satellite, shown in Fig. 1, was used as a numerical example. The substructure that was be considered for uncertainty propagation consists of the Earth pointing (+Z) reflector and tower that is mounted to the top of the bus via bars as shown in the figure.

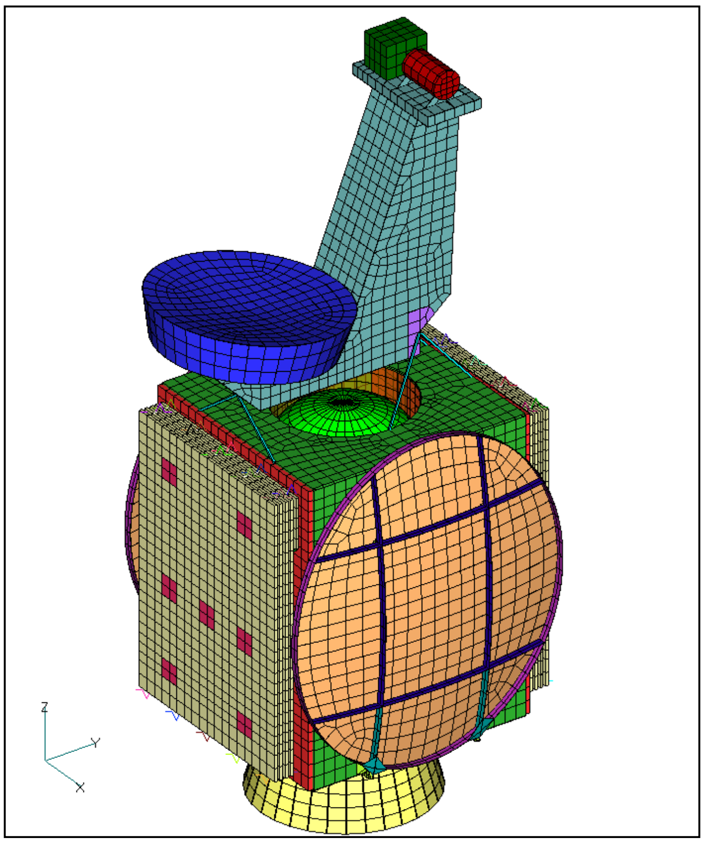

Fig. 1. Finite element model of Qsat communications satellite.

A numerical experiment was performed to investigate the probability distributions of the correlation metric terms of interest. The Maximum Entropy approach was used to randomize the nominal substructure fixed interface mass and stiffness matrices using a dispersion value of 3.0%. The corresponding modes and frequencies were computed and compared with the nominal modal parameters to yield frequency errors and cross-orthogonalities for 100,000 iterations. As expected, it was found that the uncertainties in the generalized masses, Δ_{Mjj}, frequencies, $\Delta\omega_{\mathrm{j}}$, and off-diagonal cross-orthogonalities, $\Delta\gamma_{\mathrm{ij}}$, are all zero mean and normally distributed. Figure 2 illustrates the estimate of a typical off-diagonal cross-orthogonality probability distribution.

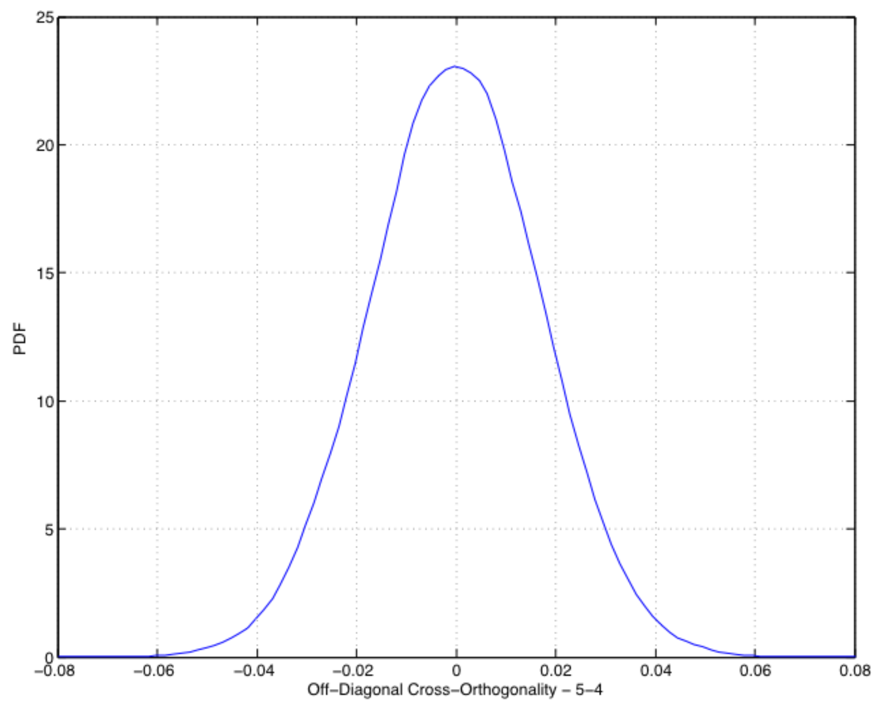

Fig. 2. Estimated PDF for cross-orthogonality term $\Delta\gamma_{5,4}$.

Figure 3 shows the estimated generalized chi-square probability distribution for the term $c_{21} = 1 - \gamma_{21,21}^2$, corresponding to free substructure mode 21. The distribution is somewhat skewed to the right because a relatively small number of off-diagonal terms in this column of the cross-orthogonality matrix contribute significantly. The theoretical details of this investigation and methods derivation can be found in Ref. [1].

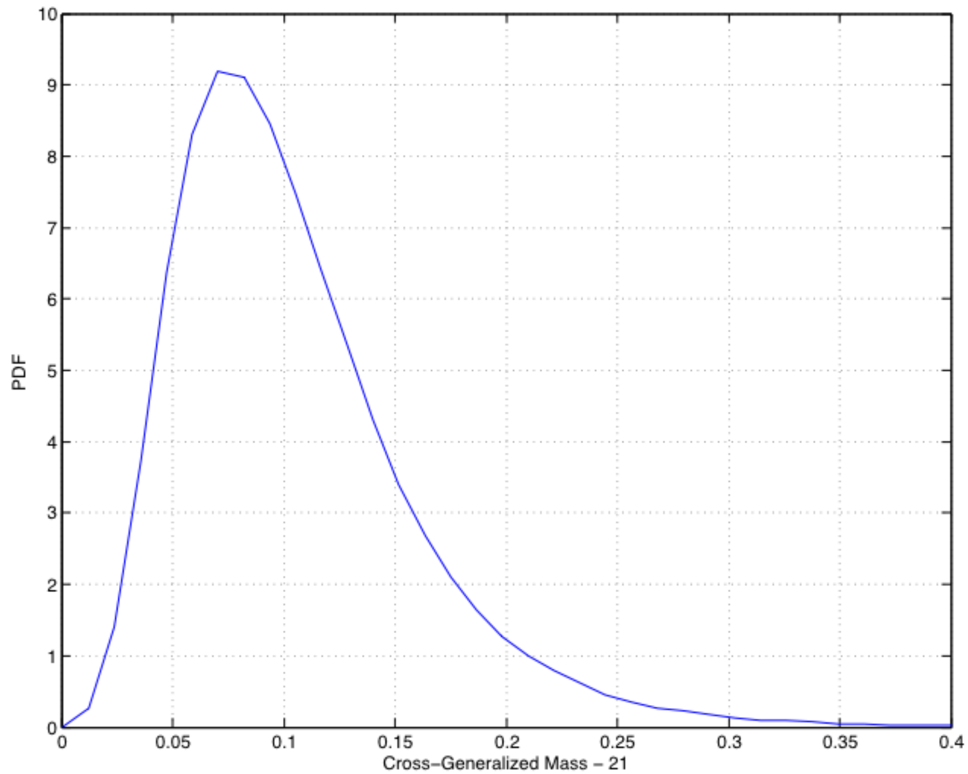

Fig. 3. Estimated PDF for $1-\gamma_{21,21}^2$.

2.2 Quantification of Uncertainty Using Frequency Response

In previous section, uncertainty was quantified in terms of test-analysis modal correlation metrics. This is plausible in the case of widely spaced modal frequencies, but it quickly breaks down in the case where there are a large number of closely spaced modes. Quantifying uncertainty directly in terms of the frequency response of the structure can circumvent the problem of closely spaced modes, or high modal density. This also makes more sense in general, because ultimately, it is the uncertainty in the response of the structure that is of importance in test-analysis correlation.

The uncertainty in the frequency response can be defined as the difference between the "truth", or test, model and the nominal FEM, and can be expressed in the form

$$\Delta H_{sai} = H_{Tsai} - H_{sai} = \phi_s \Delta h_i \phi_a^T \tag{5}$$

where Δh_i is the uncertainty in the modal frequency response relative to the nominal model modal space at frequency ω_i, and ϕ_s and ϕ_a represent the nominal modal matrix row-partitioned to the sensor and input locations, respectively. Due to variations and uncertainties in geometry, material parameters, construction, testing, etc., there is actually an ensemble of truth models. Using the fact that the product of h_i and the modal impedance matrix z_i is an identity matrix, linear perturbation can be used to relate uncertainty in modal impedance to uncertainty in physical displacement frequency response as

$$\Delta H_{ssi} = -\phi_s h_i \Delta z_i h_i \phi_s^T \tag{6}$$

It is important to note that the proposed covariance propagation approach is only as accurate as the linear perturbation relation.

In order to propagate uncertainty, Eq. (6) must be vectorized using the $vec(X)$ operator, in which the columns in matrix X are stacked column-wise. Applying this to Eq. (6) produces

$$vec(\Delta H_{ssi}) = -[\phi_s h_i \otimes \phi_s h_i] vec(\Delta z_i) \tag{7}$$

in which the symbol $\otimes$ represents the Kronecker product between two matrices. Memory requirements and computational effort can be minimized by taking advantage of the fact that both ΔH_{ssi} and Δz_i are symmetric. Defining the simplifying notation, $\Delta p_{Hi} = vec(\Delta H_{ssi})$, $\Delta p_{zi} = vec(\Delta z_i)$, and $R_i = -[\phi_s h_i \otimes \phi_s h_i]$, forming the product of Eq. (7) with its conjugate transpose *, and then taking the expectation, gives

$$E\left(\Delta p_{Hi} \Delta p_{Hi}^*\right) = C_{\Delta Hi} = R_i E\left(\Delta p_{zi} \Delta p_{zi}^*\right) R_i^* = R_i C_{\Delta zi} R_i^* \tag{8}$$

in which $C_{\Delta Hi}$ and $C_{\Delta zi}$ are the covariance matrices for uncertainty in the frequency response and modal impedance matrices at frequency ω_i, respectively. Using Eq. (8), covariance in modal impedance can be propagated into covariance in physical frequency response. Terms on the diagonal of $C_{\Delta Hi}$ are real, and represent the mean-square value of the magnitude of the individual frequency response uncertainties, $\overline{|\Delta H_{lki}|^2}$. Following the same procedure, but now taking the usual transpose results in

$$E\left(\Delta p_{Hi} \Delta p_{Hi}^T\right) = S_{\Delta Hi} = R_i E\left(\Delta p_{zi} \Delta p_{zi}^T\right) R_i^T = R_i S_{\Delta zi} R_i^T \tag{9}$$

where $S_{\Delta Hi}$ and $S_{\Delta zi}$ are symmetric matrices, termed relation matrices. Terms on the diagonal of $S_{\Delta Hi}$ are complex, and represent the mean-square value of the individual frequency response uncertainties, $\overline{\Delta H_{lki}^2}$. The complete second order statistical properties of the frequency response are specified by the two matrices $C_{\Delta Hi}$ and $S_{\Delta Hi}$. Uncertainty will be quantified at the substructure level, and then propagated into the system using the frequency response covariance and relation matrices.

Once uncertainty has been propagated into the system in the form of frequency response covariance and relation matrices, it is more convenient to recover uncertainty in magnitude and phase. Uncertainty in frequency response magnitude and phase for the lk input-output pair can be related to uncertainty in the corresponding real and imaginary parts using

$$\left\{ \begin{array}{c} \Delta|H_{lki}| \\ \Delta\theta_{lki} \end{array} \right\} = J_{lki} \left\{ \begin{array}{c} \Delta(H_{lki})_R \\ \Delta(H_{lki})_I \end{array} \right\} \tag{10}$$

where matrix J_{lki}, given by

$$J_{lki} = \begin{bmatrix} (H_{lki})_R / |H_{lki}| & (H_{lki})_I / |H_{lki}| \\ -(H_{lki})_I / |H_{lki}|^2 & (H_{lki})_R / |H_{lki}|^2 \end{bmatrix}$$

is evaluated using the nominal model at frequency ω_i, and subscripts R and I denote real and imaginary parts. Taking the expectation of the outer product gives

$$C_{pi} = \begin{bmatrix} \overline{(\Delta|H_{lki}|)^2} & \overline{(\Delta|H_{lki}|)(\Delta\theta_{lki})} \\ \therefore & \overline{(\Delta\theta_{lki})^2} \end{bmatrix} = J_{lki} \begin{bmatrix} \overline{(\Delta H_{lki})_R^2} & \overline{(\Delta H_{lki})_R (\Delta H_{lki})_I} \\ \therefore & \overline{(\Delta H_{lki})_I^2} \end{bmatrix} J_{lki}^T = J_{lki} C_{ri} J_{lki}^T \tag{11}$$

in which C_{pi} and C_{ri} are the covariance matrices for the polar and rectangular forms of the corresponding frequency response pair, respectively. The terms in C_{ri} can be recovered from the frequency response covariance $C_{\Delta Hi}$ and relation matrix $S_{\Delta Hi}$. Diagonal terms in C_{pi} represent the mean-square values of the uncertainty in the frequency response magnitude and phase. The theoretical details of this investigation and methods derivation can be found in Ref. [3].

2.3 Propagation of Substructure Uncertainty into Craig-Bampton Representation

The CB substructure representation is well suited as a building block for model validation of substructured systems. The rth substructure representation is generated using the coordinate transformation

$$u^r = \left\{ \begin{array}{c} u_o^r \\ u_a^r \end{array} \right\} = \begin{bmatrix} \phi_o^r & \psi^r \\ 0 & I \end{bmatrix} \left\{ \begin{array}{c} q^r \\ u_a^r \end{array} \right\} = T_{CB}^r u_{CB}^r \tag{12}$$

in which u_a^r represents the displacement of the substructure interface, and u_o^r is the displacement of the interior of the substructure. This representation is characterized by a combination of fixed interface substructure mode shapes, ϕ_o^r, and a set of static shapes, $\Psi^r = \begin{bmatrix} \psi^{rT} & I \end{bmatrix}^T$, called constraint modes. The substructure mass and stiffness matrices in the CB space are then given by

$$M_{CB}^r = T_{CB}^{rT} M^r T_{CB}^r \qquad K_{CB}^r = T_{CB}^{rT} K^r T_{CB}^r \tag{13}$$

where M^r and K^r are the substructure physical mass and stiffness matrices, respectively. A significant reduction in model size can be achieved by truncating the number of fixed-interface modes based on frequency.

The uncertainty in substructure modal mass can be related to the uncertainty in the CB mass matrix using

$$\Delta m^r = \phi_{CB}^{rT} \Delta M_{CB}^r \phi_{CB}^r \tag{14}$$

where ϕ_{CB}^r are the mass normalized nominal free substructure modes of interest in CB coordinates. Uncertainty in the substructure modal mass and modal stiffness are obtained from a substructure test-analysis correlation process using modal correlation metrics as described in Section 2.1. Pre- and post-multiplying each side of Eq, (14) by $M_{CB}^r \phi_{CB}^r$ and its transpose, respectively, gives

$$M_{CB}^r \phi_{CB}^r \Delta m^r \phi_{CB}^{rT} M_{CB}^r = P_T^{rT} \Delta M_{CB}^r P_T^r = \Delta M_{CBT}^r \tag{15}$$

in which matrix $P_T^r = \phi_{CB}^r \phi_{CB}^{rT} M_{CB}^r$ is an oblique projector onto the column space spanned by the nominal substructure target modes being considered in the correlation analysis. Therefore, ΔM_{CBT}^r is the uncertainty in the CB mass matrix due to the substructure target modes.

Using the vectorization procedure discussed in Section 2.2, the expression for mass uncertainty in Eq. (15) can be rewritten as

$$vec\left(\Delta M_{CBT}^r\right) = \left[R_{CBT}^r \otimes R_{CBT}^r\right] vec\left(\Delta m^r\right) \tag{16}$$

in which $R_{CBT}^r = M_{CB}^r \phi_{CB}^r$. Employing the simplifying notation, $\Delta p_{M_{CBT}}^r = vec\left(\Delta M_{CBT}^r\right)$ and $\Delta p_m^r = vec\left(\Delta m^r\right)$, Eq. (16) becomes

$$\Delta p_{M_{CBT}}^r = R^r \Delta p_m^r \tag{17}$$

Variance in the free substructure modal mass can be related to variance in the CB substructure mass using linear covariance propagation. Taking the expectation of the outer product of Eq. (17) with itself gives

$$E\left(\Delta p_{M_{CBT}}^r \Delta p_{M_{CBT}}^{rT}\right) = C_{\Delta M_{CBT}}^r = R^r E\left(\Delta p_m^r \Delta p_m^{rT}\right) R^{rT} = R^r C_{\Delta m}^r R^{rT} \tag{18}$$

in which $C_{\Delta M_{CBT}}^r$ and $C_{\Delta m}^r$ are the covariance matrices for uncertainty in CB substructure mass and free interface substructure modal mass, respectively. The analogous equation relating stiffness is given by

$$C^r_{\Delta K_{CBT}} = R^r C^r_{\Delta k} R^{rT} \tag{19}$$

The diagonal terms in the covariance matrices correspond to the mean square values, or variances, of the mass and stiffness uncertainties in the corresponding vector Δp_X. Through extensive numerical experimentation, it was found that, for sufficient uncertainty, the covariance matrices for both the modal mass and stiffness are diagonal, meaning that the terms within each of the vectors Δp^i_m and Δp^i_k are uncorrelated.

As discussed in Section 2.2, it is assumed that statistics for frequency response uncertainty are available in the form of covariance, $C^r_{\Delta Hi}$, and relation matrices, $S^r_{\Delta Hi}$, for each substructure of interest in the system. This data may come from a series of vibration tests, a database of experimental results for similar structures, analytical results based on substructure Monte Carlo (MC) analysis, etc.

In order to recover uncertainty in the substructure CB matrices quantified in terms of frequency response, a series of computations must be performed. First, the uncertainty in the *r*th substructure modal impedance must be recovered from Eq. (6). It is assumed that sensors have been placed on the substructure such that all of the nominal model modes that contribute significantly in the frequency range of interest, including rigid body modes, are linearly independent. It is also assumed that a reduced mass representation, or test-analysis model (TAM), for the substructure, M^r_{TAM}, has been developed. The TAM contains only sensor degrees of freedom. For example, the Modal, or Hybrid TAM, can be used to give an exact mass representation of the nominal substructure in the desired frequency range. If the nominal modes are mass normalized, then Eq. (6) for the *r*th substructure can be post-, and pre-multiplied by $Q^r_{si} = M^r_{TAM}\phi^r_s z^r_i$, and its transpose, respectively, to yield

$$\Delta z^r_i = -z^r_i \phi^{rT}_s M^r_{TAM} \Delta H^r_{ssi} M^r_{TAM} \phi^r_s z^r_i = Q^{rT}_{si} \Delta H^r_{ssi} Q^r_{si} \tag{20}$$

All matrices, except the uncertainties, are evaluated using the nominal FEM. The uncertainty in the *r*th substructure modal impedance can be related to the corresponding uncertainty in the CB substructure representation, ΔZ^r_{CBi}, using

$$\Delta z^r_i = \phi^{rT}_{CB} \Delta Z^r_{CBi} \phi^r_{CB} \tag{21}$$

in which ϕ^r_{CB} are the substructure modes in the CB coordinates. Equation (20) can be pre- and post-multiplied by $Q^r_{CB} = M^r_{CB}\phi^r_{CB}$ and its transpose, respectively, to produce

$$M^r_{CB} \phi^r_{CB} \Delta z^r_i \phi^{rT}_{CB} M^r_{CB} = P^{rT}_T \Delta Z^r_{CBi} P^r_T = \Delta Z^r_{CBTi} \tag{22}$$

in which the matrix $P^r_T = \phi^r_{CB}\phi^{rT}_{CB}M^r_{CB}$ is an oblique projector onto the column space spanned by the nominal substructure modes being used in the analysis. The projector spatially filters out any undesired modal contributions. Therefore, ΔZ^r_{CBTi} is the uncertainty in the CB impedance matrix due to the substructure modes that are dynamically important in the frequency range of interest. Equation (20) then yields

$$\Delta Z^{r}_{CBTi} = Q^{r}_{CB} Q^{rT}_{si} \Delta H^{r}_{ssi} Q^{r}_{si} Q^{rT}_{CB} \tag{23}$$

or using the vectorization procedure from the previous section

$$\Delta p^{r}_{Z_{CBTi}} = vec\left(\Delta Z^{r}_{CBTi}\right) = -\left[Q^{r}_{CB} Q^{rT}_{si} \otimes Q^{r}_{CB} Q^{rT}_{si}\right] vec\left(\Delta H^{r}_{ssi}\right) = -R^{r}_{CBsi} \Delta p^{r}_{Hi} \tag{24}$$

Following the process outlined previously, the covariance and relation matrices for the uncertainty in the rth substructure CB impedance matrix can be recovered from the corresponding substructure frequency response uncertainties.

Quantifying the frequency response uncertainty in terms of the substructure CB impedance matrices, and then propagating into the system using CMS, has a significant advantage over methods that use direct assembly of substructure frequency response, in that translations and rotations at substructure interfaces do not have to be measured. The theoretical details of this investigation and methods derivation can be found in Refs. [2] and [3].

2.4 Propagation of Substructure Uncertainties into System Level Matrices

The system matrices can be synthesized from the CB substructure representations by applying the appropriate constraints at the substructure interfaces. The uncoupled system displacement vector u_G can be related to the coupled system displacement vector u_{SYS} using the transformation

$$u_G = \begin{Bmatrix} u^{1}_{CB} \\ u^{2}_{CB} \\ \vdots \\ u^{n_{sub}}_{CB} \end{Bmatrix} = \begin{bmatrix} T^{1} \\ T^{2} \\ \vdots \\ T^{n_{sub}} \end{bmatrix} \begin{Bmatrix} u_Q \\ u_A \end{Bmatrix} = T u_{SYS} \tag{28}$$

where

$$u_Q = \begin{Bmatrix} q^{1T} & q^{2T} & \cdots & q^{n_{sub}T} \end{Bmatrix}^{T}$$

is a partition of the coupled system displacement containing all of the component modal degrees of freedom from the substructures, and u_A is the partition containing all the non-redundant substructure interface degrees of freedom.

Uncertainty in the uncoupled CB substructure mass matrices can be related to the uncertainty in the system coupled modal mass matrix using the expression

$$\Delta \mathscr{M}_{SYS} = \phi^{T}_{SYS} T^{T} \Delta M_G T \phi_{SYS} = \sum_{i=1}^{n_{sub}} \phi^{T}_{SYS} T^{rT} \Delta M^{r}_{CBT} T^{r} \phi_{SYS} \tag{29}$$

where ϕ_{SYS} are the system modes of interest. Using the procedure outlined in the previous sections for substructures, the system modal mass covariance matrix can be written in the compact form

$$C_{\Delta \mathscr{M}_{SYS}} = \sum_{i=1}^{n_{sub}} W^r C^r_{\Delta M_{CBT}} W^{rT} \tag{30}$$

in which matrices $W^r = \left[\phi^T_{SYS} T^{rT} \otimes \phi^T_{SYS} T^{rT} \right]$, and it is assumed that the individual substructure uncertainties are uncorrelated with one another. A parallel analysis produces the same form for the system modal stiffness uncertainty

$$C_{\Delta \mathscr{K}_{SYS}} = \sum_{i=1}^{n_{sub}} W^r C^r_{\Delta K_{CBT}} W^{rT} \tag{31}$$

The mean square values of the uncertainties in the system modal mass and stiffness matrices can then be recovered from the diagonal of the matrices $C_{\Delta \mathscr{M}_{SYS}}$ and $C_{\Delta \mathscr{K}_{SYS}}$, respectively. Once the uncertainty in the system modal mass and stiffness is recovered, the uncertainty in the system modal correlation metrics can be determined using the methods presented in Section 2.1. It was shown that the variance of the off-diagonal system cross-orthogonality terms can be expressed as

$$E\left(\Delta \gamma^2_{SYSij}\right) = \frac{1}{\left(\Omega_{SYSi} - \Omega_{SYSj}\right)^2} \left[E\left(\Delta \mathscr{K}^2_{SYSij}\right) + \Omega^2_{SYSj} E\left(\Delta \mathscr{M}^2_{SYSij}\right) \right] \tag{32}$$

where Ω_{SYSi} is the ith system eigenvalue. The variance of the system natural frequencies is given by

$$E\left(\Delta \omega^2_{SYSj}\right) = \frac{1}{4\Omega_{SYSj}} \left[E\left(\Delta \mathscr{K}^2_{SYSjj}\right) + \Omega^2_{SYSj} E\left(\Delta \mathscr{M}^2_{SYSjj}\right) \right] \tag{33}$$

Following the same procedure, the system modal impedance covariance matrix is then given by

$$C_{\Delta \mathscr{Z}_{SYSi}} = \sum_{r=1}^{n_{sub}} W^r C^r_{\Delta Z_{CBTi}} W^{rT} \tag{34}$$

The corresponding system modal impedance relation matrix is then

$$S_{\Delta \mathscr{Z}_{SYSi}} = \sum_{r=1}^{n_{sub}} W^r S^r_{\Delta Z_{CBTi}} W^{rT} \tag{35}$$

The uncertainty in the physical system frequency response at the desired n_{sSYS} output locations due to n_{aSYS} selected input locations is

$$\Delta H_{SYSsai} = -T_{sP} \mathscr{H}_{SYSi} \Delta \mathscr{Z}_{SYSi} \mathscr{H}_{SYSi} T^T_{aP} \tag{36}$$

where T_{sP} is the transformation from system modal coordinates p to the selected output locations and T_{aP} is the corresponding transformation to the physical input locations. The covariance matrix for the system physical frequency response can then be recovered from the system modal impedance covariance matrix using

$$C_{\Delta H_{SYSi}} = Y_i C_{\Delta \mathcal{Z}_{SYSi}} Y_i^* \tag{37}$$

The corresponding system frequency response relation matrix is then

$$S_{\Delta H_{SYSi}} = Y_i S_{\Delta \mathcal{Z}_{SYSi}} Y_i^T \tag{38}$$

Uncertainty in system frequency response magnitude and phase can then be recovered the results from Section 2.2.

A simple example is considered to illustrate the application of the proposed uncertainty propagation technique. The system consists of two steel beam substructures attached in the shape of a T. Both substructures are constrained to plane motion. For illustration, only substructure 2 is assumed to have uncertainty. The uncertainty is quantified in terms of test-analysis modal correlation metrics discussed in Section 2.1. In this example, it is assumed that correlation uncertainty is available for the first 20 elastic modes of substructure 2. The free-free vibration test results were simulated using Monte Carlo analysis. At each iteration, the nominal physical mass and stiffness matrices for substructure 2 were randomized using the Maximum Entropy approach. In this example, a dispersion level of 15% was selected for both mass and stiffness, and 10,000 iterations were performed. The resulting test-analysis correlation statistics for substructure 2 in the form of RMS cross-orthogonality matrix is illustrated in Fig. 4. The 5th percentile for cross-generalized mass, $\gamma_{jj0.95}$, indicates that modes 10, 11, 15, 16, 19, and 20 do not pass the AF criterion for cross-orthogonality, $\gamma_{jj} \geq 0.95$, with 95% confidence. Maximum one-sigma off-diagonal cross-orthogonality terms indicate that only modes 1 through 8 pass the AF criterion $\gamma_{ij} \leq 0.10$ with 95% confidence. Taking the intersections of the results, only the first 8 elastic modes for random substructure 2 strictly pass all of the AF correlation criteria at the 95% confidence level. Uncertainties in the modal mass and stiffness for the first 20 elastic modes of substructure 2 are recovered from the test-analysis correlation uncertainty. Figure 5 shows the recovered RMS modal mass uncertainty. The uncertainties in the substructure 2 modal matrices were then propagated into uncertainties in the corresponding CB representation, and then into the modal mass and stiffness uncertainties for the first 20 elastic system modes. Figure 6 shows the corresponding recovered RMS system cross-orthogonality matrix. Correlation statistics predicted using a Monte Carlo analysis validated the accuracy of the proposed covariance propagation approach. Both methods predict that only the first 10 system modes strictly pass all of the AF correlation at the 95% confidence level. The theoretical details of this investigation and methods derivation can be found in Ref. [2].

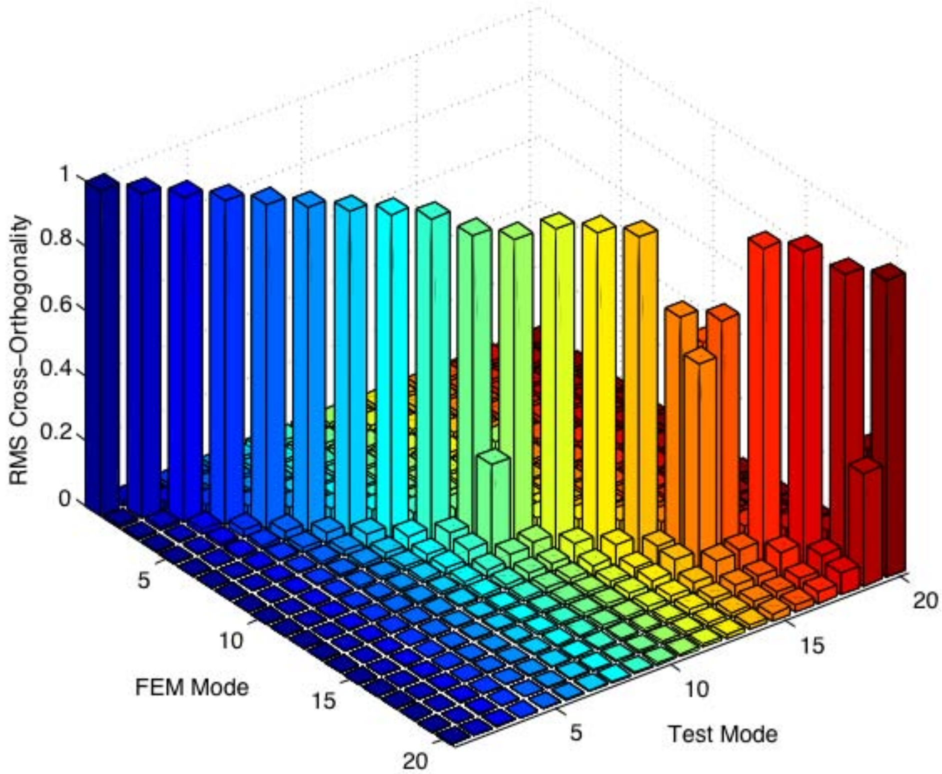

Fig. 4. Substructure 2 RMS cross-orthogonality.

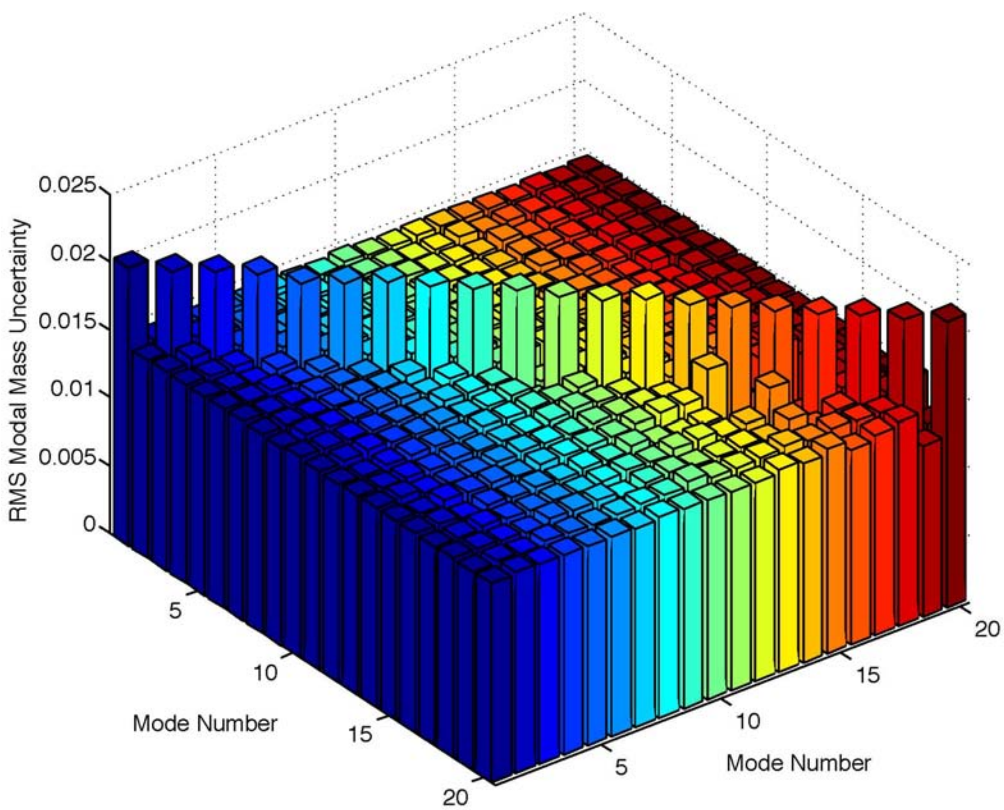

Fig. 5. Substructure 2 rms modal mass uncertainty.

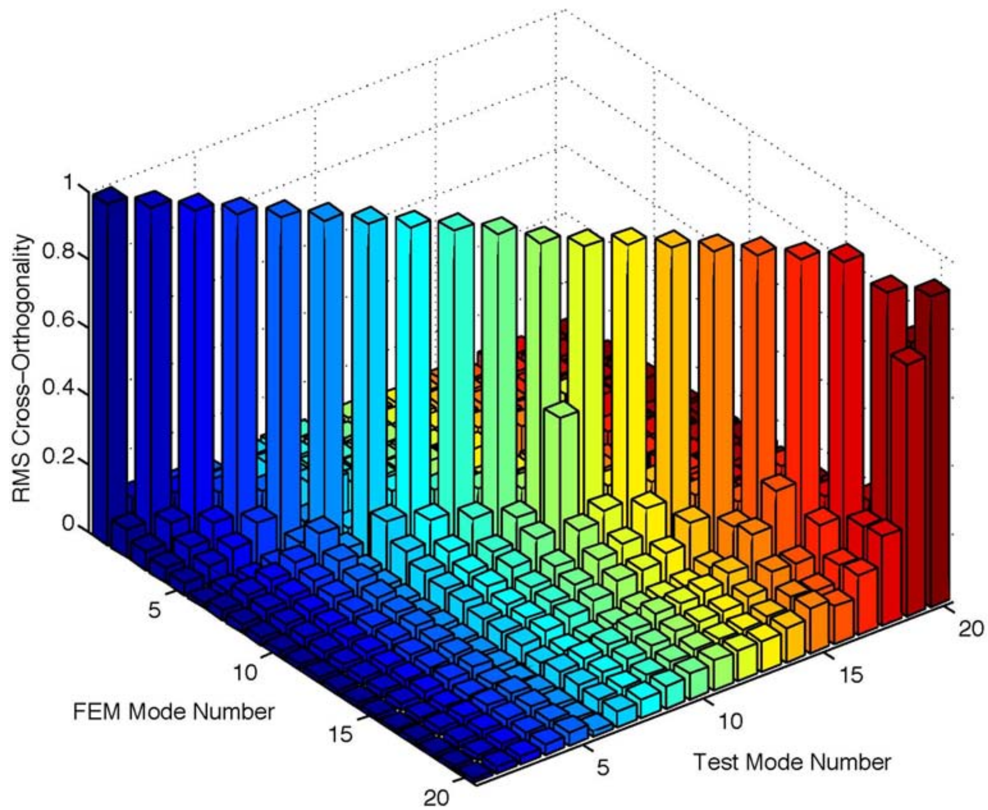

Fig. 6. System RMS cross-orthogonality for first 20 elastic modes.

The same example is used to illustrate the propagation of the uncertainty in the substructure-2 velocity frequency response, that is spanned by the first 20 nominal elastic modes, into the system velocity frequency response. Covariance and relation matrices for the substructure-2 impedance matrix in CB coordinates were derived at 200 evenly spaced frequencies from 0.0 to 6000.0 Hz. Using the proposed CMS based covariance propagation procedure, the system modal impedance covariance and relation matrices were then recovered. Next, the associated covariance and relation matrices for system velocity frequency response were derived, and then finally uncertainty in system velocity frequency response magnitude and phase were predicted. The accuracy of the proposed covariance propagation procedure is demonstrated by comparing propagation results with a corresponding 10,000 iteration system MC analysis.

Figure 7 compares the standard deviation of a system level drive point velocity frequency response magnitude predicted using the covariance propagation analysis with the results of a full MC analysis. Figure 8 shows the corresponding for standard deviation in phase. In both cases, the results predicted using covariance propagation agree very well with the results predicted using a full MC analysis at both isolated resonances, and in regions of closely spaced mode pairs. It was found in previous work performed during this investigation that uncertainty propagation using modal correlation metrics performed very poorly for these mode pairs. The magnitude of a nominal model velocity drive point frequency response is shown in Fig. 9. In addition, the 95th

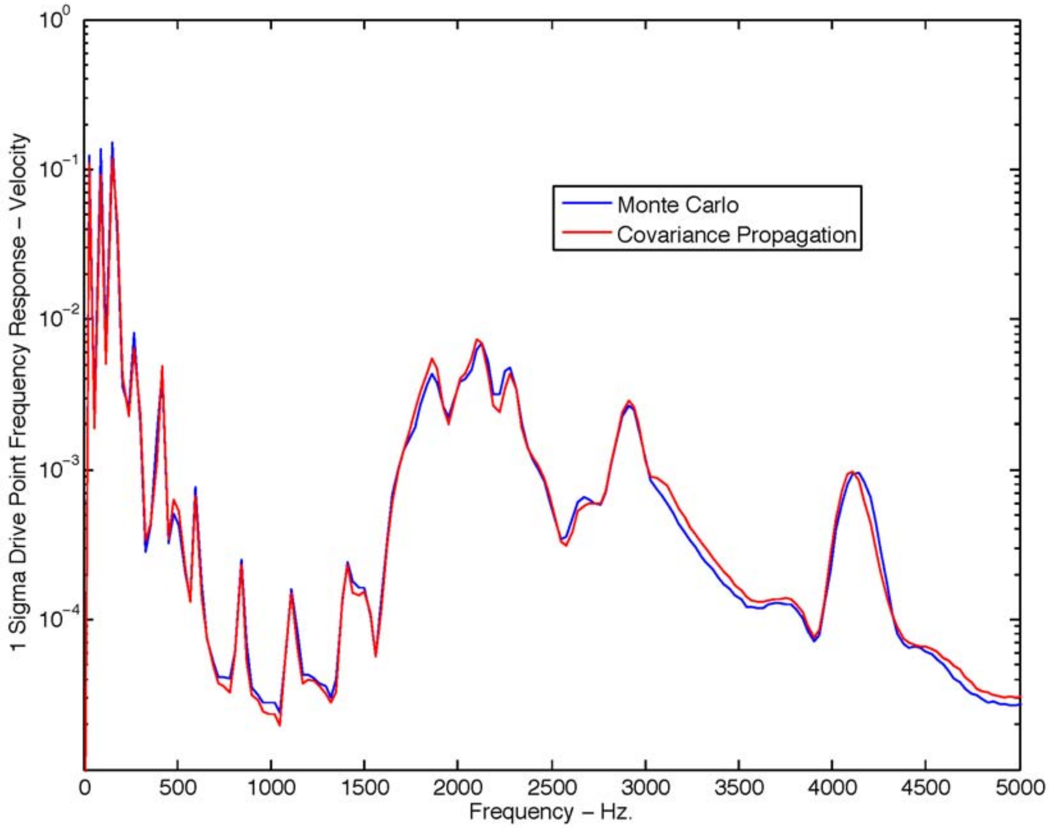

Fig. 7. Standard deviation of drive-point velocity frequency response magnitude at dof 301x.

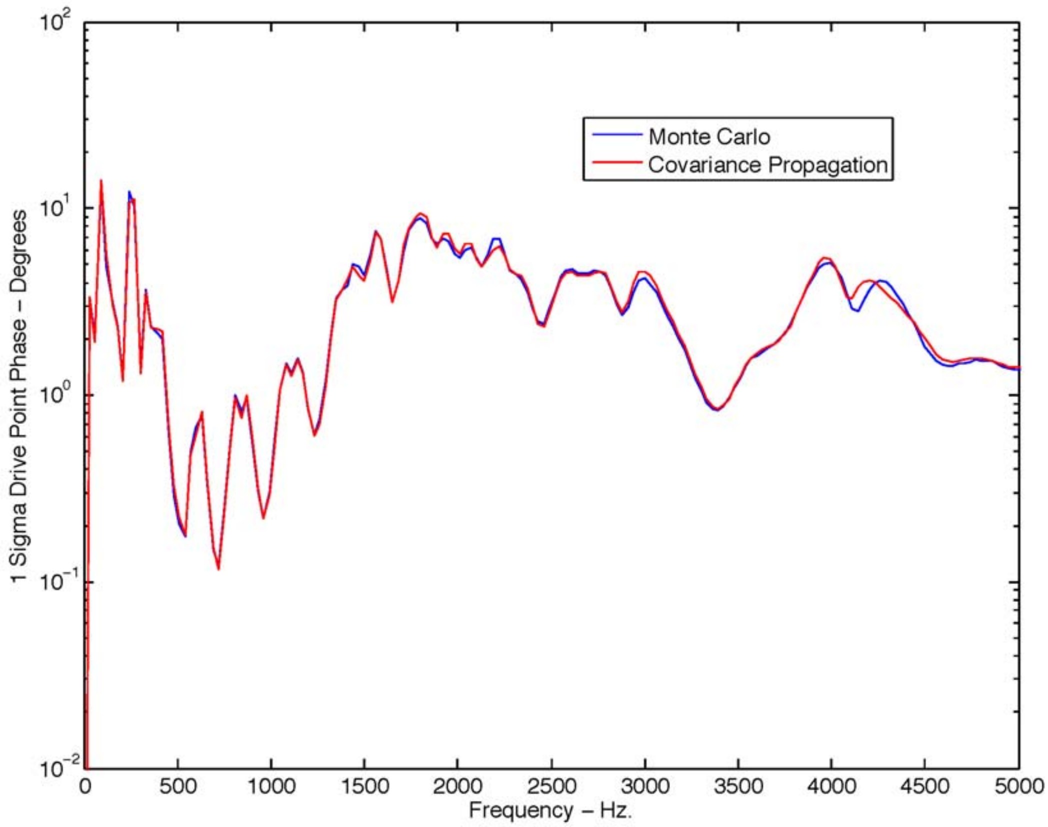

Fig. 8. Standard deviation of drive-point velocity frequency response phase at dof 301x.

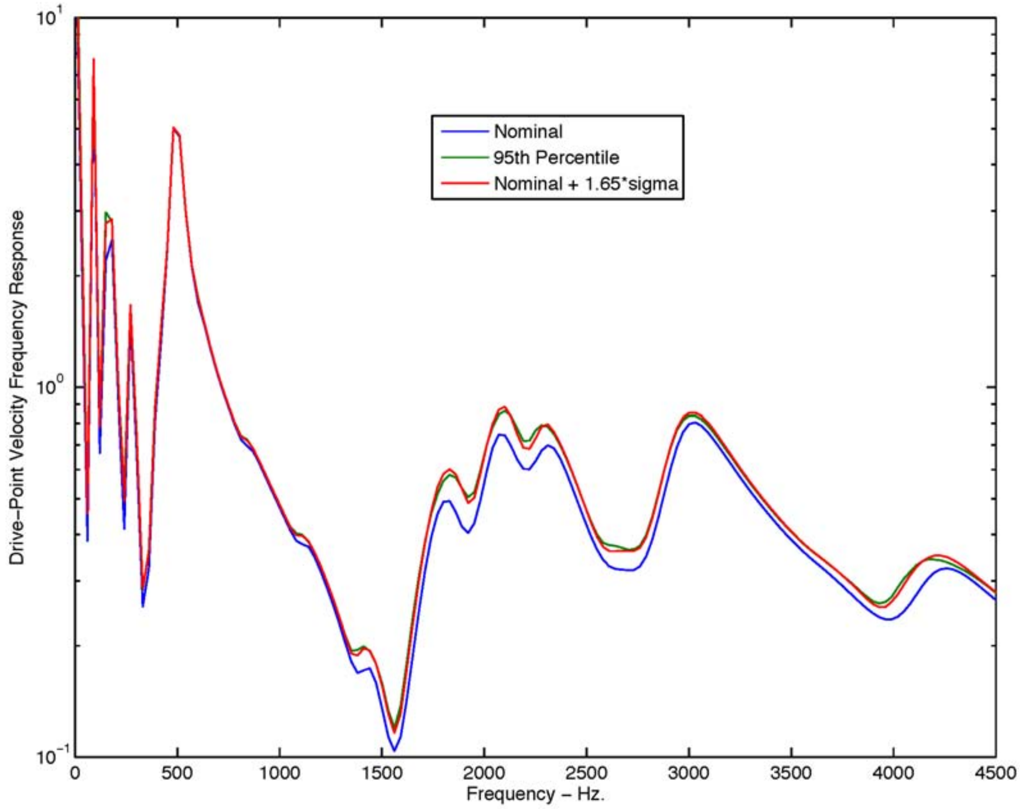

Fig. 9. Drive-point velocity frequency response magnitude at dof 301x.

percentile from the MC simulation, and the nominal response plus 1.65 standard deviations predicted by covariance propagation are illustrated. The close agreement of these last two curves indicates that for this particular case, the magnitude of the velocity frequency response is normally distributed. The MC analysis validates the accuracy of the proposed frequency response uncertainty propagation procedure. However, the propagation approach offers a significant computation time advantage over the MC analysis. The theoretical details of this investigation and methods derivation can be found in Ref. [3].

3.0 CONCLUSION

This research project has produced a complete and systematic procedure for studying the effects of substructure uncertainty on the test-analysis correlation of complex spacecraft that are validated on a substructure-by-substructure basis, using test and analysis comparisons. The uncertainty is quantified in terms of accepted modal test-analysis correlation metrics, and covariance and relation matrices associated with the differences in the test and FEM frequency response. Linear perturbation analysis is used to relate uncertainty in correlation metrics to uncertainty in substructure matrices. Covariance propagation is then used to propagate substructure uncertainty into the expected correlation metric uncertainty for the system using a Craig-Bampton based component mode synthesis approach. The frequency response based procedure is of special interest in systems with high modal density, where modal methods do not work. The results of this research are significant to the Air Force because critical decisions concerning space structure performance and survivability are made based on the results of test-analysis correlation. In many situations, it is either impossible to perform a system vibration test, or it is highly desirable to avoid one to conserve time and resources. If modeling and analysis are to replace system tests, then it is imperative to have confidence in the results. Correlation/validation is the path to providing this confidence and determining the predictability of models used in the decision making process. Within this new validation paradigm, there is no system level test data available. Therefore, a probabilistic system correlation must be performed by quantifying uncertainty in the system's substructures, and then propagating it into the system correlation metrics. An understanding of substructure correlation requirements will positively impact the speed of the loads analysis process, and reduce time to space.

4.0 REFERENCES

1. Kammer, D. C., and Nimityongskul, S., "Propagation of Uncertainty in Test-Analysis Correlation of Substructured Spacecraft," *Journal of Sound and Vibration*, Vol. 330, No. 6, 1211-1224, 2010.

2. Kammer, D. C., and Krattiger, D., "Propagation of Spacecraft Free-Interface Substructure Uncertainty into System Test-Analysis Correlation," *Journal of Vibration and Acoustics*, Vol. 134, No. 2, Oct. 2012, 051014.

3. Kammer, D. C., and Krattiger, D., "Propagation of Uncertainty in Substructured Spacecraft using Frequency Response," accepted *AIAA Journal*, Jul. 2012.

4. Nimityongskul, A.P., Kammer, D.C., Lacy, S., and Babuska, V., "Frequency Domain Test-Analysis Correlation in the Presence of Model Uncertainty," Structural Dynamics, Vol. 3, Conference Proceedings of the Society of Experimental Mechanics Series, Vol. 12, pp. 403-418, 2011.

APPENDIX

This Appendix contains listings of the main Matlab computer algorithms developed during the course of this project. Documentation on the use of the computer codes is listed in the files themselves. Table A-1 lists function “recover”. This function recovers correlation metric variances. Table A-2 lists function “xcovmatfull”. This function generates modal matrix covariance matrices. Table A-3 lists function “xrandcbKM2”. This function uses random matrix theory to generate random systems and corresponding statistics. Table A-4 lists function “xrecovermag”. This function recovers uncertainty in frequency response magnitude and phase.

Table A-1. Function “recover” recovers correlation metric variances.

```
function [rmsw,rmsCO,CGM05] = recover(msM,msK,L);
%
%   Created by:     Daniel C. Kammer
%                   Professor
%                   Dept of Engineering Physics
%                   University of Wisconsin
%                   Madison, WI  53706
%                   (608) 262-5724
%
%
%  ==================================================================
%
% This function recovers the correlation metric variances from the
% mean square modal mass and stiffness uncertanties:
%
%
%
%  HISTORY
%  =======
%
%    Created:  Daniel C. Kammer       1-09-10
%
%    Updated:
%               3-27-10   DCK    corrections made to frequency uncertainty
%                                recovery
%
%  ========================================================================
%
%  INPUT
%  =====
%  L       =  elastic modal eigenvalues  -  n x 1
%
%  msM     = mean square modal mass uncertainty  -  n x n
%
%  msK     = mean square modal stiffness uncertainty  -  n x n
%
%  OUTPUT
%  ======
%
%  rmsw   = rms frequency uncertainty  -  n x 1
%
```

```
%  rmsCO = rms Cross-Ortho  -  n x n
%
%  CGM05 = 95th percentile of CGM assuming a generalized chi-square dist
%
%  Use:  [rmsw,rmsCO] = recover(msM,msK,L);
%======================================================================
%
Date=date
%
n = size(L,1);
nb = n*(n+1)/2;
jj = 0;
%
rmsw = zeros(n,1);                                                      % designate space
rmsCO = zeros(n,n);                                                     % designate space
msCO = zeros(n,n);                                                     % designate space
CGM05 = zeros(n,1);                                                     % designte space
%
for j = 1:n
    x = zeros(100000,1);                                                % designate space
    for i = 1:n

            if i == j

                rmsw(i) = sqrt((msK(i,i) + (L(i)^2)*msM(i,i))/(4*L(i))); % rms freq

            else

                msCO(i,j) = (msK(i,j) + (L(j)^2)*msM(i,j))/(L(i)-L(j))^2; % ms off-diag

                x = x + msCO(i,j)*chi2rnd(1,100000,1);

            end
    end

    msCO(j,j) = 1 - sum(msCO(:,j));                     % CGM
    x = sort(x);
    CGM05(j) = sqrt(1-x(95000));

end

rmsCO = sqrt(msCO);                                     % rms CO
%rmsCO = (msCO);
end
```

Table A-2. Function “xcovmatfull” generates modal matrix covariance matrices.

```
function [Cm,Ck] = xcovmatfull(L,stdw,stdm,rmsCO);
%
%   Created by:     Daniel C. Kammer
%                   Professor
%                   Dept of Engineering Physics
%                   University of Wisconsin
%                   Madison, WI  53706
%                   (608) 262-5724
%
%  =================================================================
%
% This function generates the modal mass and stiffness covariance matrices
% based on the uncertainty matrices stacked by column and then truncated to
% the lower triangular partition:
%
%     vech(deltam) and vech(deltak)
%
%  Covariance matrices are based on user provided rms uncertainties in
%  frequencies, generalized masses, and cross-orthogonality
%
%  Allows user to generate covariance matrices for a smaller number of
%  modes than frequencies in L, and then the matrices are expanded to size
%  of modes in L using zeros.  This allows the user to use the same
%  transformation matrix in covariance propagation, but trying various
%  numbers of modes in the propagation.
%
%  HISTORY
%  =======
%
%    Created:  Daniel C. Kammer       07-27-10
%
%    Modified:
%
%  =======================================================================
%
%  INPUT
%  =====
%  L       =  elastic nominal modal eigenvalues  -  n x 1
%
%  stdw    =  standard deviation of frequencies in percent  - n x 1
%
```

```
%  stdm =  standard deviation in modal masses  -  n x 1
%
%  rmsCO   =  rms cross-orthogonality matrix  -  n x n
%
%
%  OUTPUT
%  ======
%
%  Cm  = modal mass covariance matrix  -  [n(n+1)/2] x [n(n+1)/2]
%
%  Ck = modal stiffness covariance matrix  -  [n(n+1)/2] x [n(n+1)/2]
%
%
%  Use:  [Cm,Ck] = xcovmatfull(L,stdw,stddelm,rmsCO);
%=======================================================================
%
Date=date
%
nr = input('Number of Modes to Propagate  -  ')               % prompt for number of modes to compute cov
%
n = size(L,1);
nb = nr*(nr+1)/2;
jj = 0;
%
w = sqrt(L);                                                            % nominal natural frequencies
%
stdw = stdw/100;
%
s = rmsCO.^2;                                                  % variance of crodd-ortho
%
Cm = zeros(nb,1);                                                      % initialize mass covariance matrix
Ck = zeros(nb,1);                                                      % initialize stiffness covariance
matrix
%
%  Build Covariance Matrices
%  -------------------------

for j = 1:nr

    for i = 1:nr

        if i >= j
            jj = jj+1;
```

```
            if i == j

                Cm(jj) = stdm(j)^2;                                        % diagonal mass variance
                Ck(jj) = 4*L(j)*(stdw(j)^2)*L(j) - (L(j)^2)*stdm(j)^2;        % diagonal stiffness variance


            else
                GijGji = -[L(j)*s(j,i) + L(i)*s(i,j)]/[L(i)+L(j)];   % mean value of Gij*Gji
                Cm(jj) = s(j,i) + s(i,j) + 2*GijGji;                  % off-diag mass variance
                Ck(jj) = (L(j)^2)*s(j,i) + (L(i)^2)*s(i,j) + 2*L(i)*L(j)*GijGji; % off-diag stiffness var

            end
        end
    end
end


if n > nr

   [Sn,Sni] = dupmat(n);
   [Snr,Snri] = dupmat(nr);
   Cm = Snri * Cm;                                                  % expand to nr^2 size vector
   Ck = Snri * Ck;                                                  % expand to nr^2 size vector

   msm = reshape(Cm,nr,nr);                                         % ms dm
   msk = reshape(Ck,nr,nr);                                         % ms dk

   msm = [msm zeros(nr,n-nr); zeros(n-nr,n)];                       % expand to nxn
   msk = [msk zeros(nr,n-nr); zeros(n-nr,n)];                       % expand to nxn

   Cm = Sn * msm(:);                                                % transform to vech
   Ck = Sn * msk(:);                                                % transform to vech

end

Cm = diag(Cm);
Ck = diag(Ck);

end
```

```
function [Sn,Sni] = dupmat(n);
%
%   Created by:     Daniel C. Kammer
%                   Professor
%                   Dept of Engineering Physics
%                   University of Wisconsin
%                   Madison, WI  53706
%                   (608) 262-5724
%
%  =================================================================
%
% This function generates "Duplication Matrix" Sn and its inverse Sni
% such that:
%
%      vech(X) = Sn * vec(X)
%      vec(X) = Sni * vech(X)
%
%  where X is a square symmetric matrix and vech(X) is the vectorized lower
%  triangular partition of X, by column.
%
%
%  HISTORY
%  =======
%
%    Created:  Daniel C. Kammer       12-4-09
%
%  =======================================================================
%
%  INPUT
%  =====
%  n        =  dimension of X
%
%  OUTPUT
%  ======
%
%  Sn   = nth order duplication matrix  -  [n(n+1)/2] x n^2
%
%  Sni = inverse of nth order duplication matrix  -  n^2 x [n(n+1)/2]
%
%
%
%  Use:  [Sn,Sni] = dupmat(n);
%=======================================================================
%
```

```
Date=date
%
nb = n*(n+1)/2;
n2 = n^2;
%
in = [1:n2]';
in = reshape (in,n,n);                    % transform "in" to nxn
%
S = eye(n2,n2);
%
jj = 0;
kk = 0;
ia = zeros(1,nb);
%
for i = 1:n

    for j = 1:n

        if j >= i
            jj = jj+1;
            ia(jj) = in(j,i);
            if j > i
                S(:,in(j,i)) = S(:,in(j,i)) + S(:,in(i,j));
            end
        end
    end
end

Sn = eye(n2);
Sn = Sn(ia,:);

Sni = S(:,ia);

clear S

end
```

Table A-3. Function “xrandcbKM2” uses random matrix theory to generate random systems and corresponding statistics.

```
function
[Cdm,Cdk,CdKPHI,CdMPHI,dvarm,dvark,dvarKcb,dvarMcb,dvarKPHI,dvarMPHI,dL,Lr,CGMr]=xrandcbKM2(K,M,Tcb,io,ia);
%
%    Created by:      Daniel C. Kammer
%                     Professor
%                     Dept of Engineering Physics
%                     University of Wisconsin
%                     Madison, WI  53706
%                     (608) 262-5724
%
%
%   =================================================================
%
%   This program uses random matrix theory to generate random
%   stiffness and mass matrices and statistics for propagation of stiffness
%   uncertainty subject to the constraints that the random matrices
%   are symmetric and positive semidefinite.
%
%   It is assumed that the stiffness matrix has nr rigid
%   body modes.  Statistics are derived for the full system,the CB
%   representation, and the fixed system.
%
%   Soize, "Random Matrix Theory for Modeling Uncertainties in
%   Computational Mechanics," Comp. Meth. Appl. Mech. and Eng., 2004.
%
%
%   HISTORY
%   =======
%
%     Created:  Daniel C. Kammer       03-25-10
%
%   =======================================================================
%
%   INPUT
%   =====
%   K        =  nominal stiffness matrix
%
%   OUTPUT
%   ======
%
```

```
%  CdK = stiffness uncertainty covariance matrix
%  dKr = vectorized random stiffness by iteration
%
%  Use:  [CdK,dKr]=xrandK(K);
%=======================================================================
%
Date=date
%
nit = input('Number of Iterations -')  % prompt for number of samples in ensemble
%
nr = input('Number of Rigid Body Modes -')  % prompt for number of rigid body modes
%

delm = input('Dispersion Level for Mass (%) -')  % prompt for dispersion level for mass matrix
delk = input('Dispersion Level for Stiffness (%) -')  % prompt for dispersion level for mass matrix

delm = delm/100;
delk = delk/100;

nm = size(K,1);                               % size of stiffness matrix
no = size(io,1);
n2 = nm^2;
ne = nm - nr;                                 % number of elastic modes
ncb = size(Tcb,2);
na = size(ia,1);
nq = ncb-na;

mm = fix((no+1)/delm^2);                      % number of random vectors for mass matrix
mk = fix((ne+1)/delk^2);                      % number of random vectors for stiffness matrix

%

dKr = zeros(n2,nit);
dKrcb = zeros(ncb^2,nit);
dMr = zeros(n2,nit);
dMrcb = zeros(ncb^2,nit);
dMrPHI = zeros(ncb^2,nit);
dKrPHI = zeros(ncb^2,nit);
dmr = zeros(nq^2,nit);
dkr = zeros(nq^2,nit);
CGMr = zeros(ncb^2,nit);

Lr = zeros(nm,nit);
Lrcb = zeros(ncb,nit);
```

```
Lrc = zeros(nq,nit);
dLcb = zeros(ncb,nit);
dLc = zeros(nq,nit);
dL = zeros(nm,nit);

Ms = [M(io,io) M(io,ia); M(ia,io) M(ia,ia)];  % resort mass matrix
Ks = [K(io,io) K(io,ia); K(ia,io) K(ia,ia)];  % resort stiffness matrix

Mcb = Tcb'*Ms*Tcb;                            % nominal mass in CB coords
Kcb = Tcb'*Ks*Tcb;                            % nominal stiffness in CB coords

%
%  Compute Nominal Modes of Ks
%  ---------------------------

[PHI,L] = eig(Ks);
[L,i] = sort(diag(L));                        % sort in ascending order
PHI = PHI(:,i);                               % resort modes
PHI = PHI*diag(diag(PHI'*PHI).^(-.5));        % normalize to unit length
Le = L(nr+1:nm);                                 % elastic mode eigenvalues
Pe = PHI(:,nr+1:nm);                             % elastic modes
Lnm = diag(Le.^.5)*Pe';                       % decomposition:  Kb = Lnm'*Lnm

Moo = M(io,io);
Koo = K(io,io);

Lm = chol(Moo);                               % Cholesky factorization of M

Lc = sort(eig(Koo,Moo));                      % nominal fixed eigenvalues
Lc = Lc(1:nq);

[PHIcb,Lcb] = eig(Kcb,Mcb);                   % nominal CB eigenvalues and modes
[Lcb,i] = sort(diag(Lcb));                    % sort in ascending order
PHIcb = PHIcb(:,i);                           % resort modes
PHIcb = PHIcb*diag(diag(PHIcb'*Mcb*PHIcb).^(-.5));        % normalize to unit mass


L = sort(eig(K,M));                           % nominal free eigenvalues for full system

for i = 1:nit
```

```
%  Randomize stiffness matrix
%  --------------------------

    x = randn(ne,mk);                             % normal random vectors with zero mean and unit variance

    Kr = Lnm'*(x*x')*Lnm/mk;                      % physical coordinates
    dK = Kr - Ks;

    dKr(:,i) = dK(:);                             % vectorized ith random stiffness uncertainty

    Kr = Ks + dK;

%  Randomize mass matrix
%  ---------------------

    x = randn(no,mm);                             % normal random vector with zero mean and unit variance

    mr = Lm'*(x*x')*Lm/mm;
    Mr = [mr M(io,ia); M(ia,io) M(ia,ia)];        % sorted random mass matrix

    dM = Mr - Ms;                                 % sorted mass error
    dMr(:,i) = dM(:);                             % vectorized ith random stiffness uncertainty

    Mrcb = Tcb'*Mr*Tcb;                           % CB mass
    dMcb = Tcb'*dM*Tcb;                           % mass uncertainty in CB space
    dMrcb(:,i) = dMcb(:);
    dMPHI = PHIcb'*dMcb*PHIcb;                    % CB modal mass uncertainty
    dMrPHI(:,i) = dMPHI(:);
    dm = dMcb(1:nq,1:nq);                         % fixed mode mass uncertainty
    dmr(:,i) = dm(:);

    Krcb = Tcb'*Kr*Tcb;
    dKcb = Tcb'*dK*Tcb;
    dKrcb(:,i) = dKcb(:);
    dKPHI = PHIcb'*dKcb*PHIcb;                    % CB modal stiffness uncertainty
    dKrPHI(:,i) = dKPHI(:);
    dk = dKcb(1:nq,1:nq);                         % fixed mode stiffness uncertainty
    dkr(:,i) = dk(:);

    kr = Krcb(1:nq,1:nq);                         % fixed interface modal stiffness
    mr = Mrcb(1:nq,1:nq);                         % fixed interface modal mass

    Lr(:,i) = sort(eig(Kr,Mr));                   % randomized eigenvalues of full substructure
    dL(:,i) = Lr(:,i) - L;                        % uncertrainty in full system eigenvalues
```

```
    Lrcb(:,i) = sort(eig(Krcb,Mrcb));          % randomized eigenvalues of CB representation
    dLcb(:,i) = Lrcb(:,i) - Lcb;               % uncertrainty in CB system eigenvalues

    Lrc(:,i) = sort(eig(kr,mr));               % randomized eigenvalues of fixed representation
    dLc(:,i) = Lrc(:,i) - Lc;                  % uncertrainty in fixed system eigenvalues

    [PHIcbr,Lcbr] = eig(Krcb,Mrcb);            % random CB eigenvalues and modes
    [Lcbr,iis] = sort(diag(Lcbr));             % sort in ascending order
    PHIcbr = PHIcbr(:,iis);                    % resort modes
    PHIcbr = PHIcbr*diag(diag(PHIcbr'*Mcb*PHIcbr).^(-.5));        % normalize to unit nominal mass

    CGM = PHIcb'*Mcb*PHIcbr;                   % cross ortho
    CGMr(:,i) = CGM(:);                        % vectorize

end

CdKcb = cov(dKrcb');                           % covariance matrix for uncertainty in CB stiffness
dvarKcb = reshape(diag(CdKcb),ncb,ncb);        % variance of CB stiffness uncertainty

CdMcb = cov(dMrcb');                           % covariance matrix for uncertainty in CB mass
dvarMcb = reshape(diag(CdMcb),ncb,ncb);        % variance of CB stiffness uncertainty

CdKPHI = cov(dKrPHI');                         % covariance matrix for uncertainty in CB modal stiffness
dvarKPHI = reshape(diag(CdKPHI),ncb,ncb);       % variance of CB modal stiffness uncertainty

CdMPHI = cov(dMrPHI');                         % covariance matrix for uncertainty in CB modal mass
dvarMPHI = reshape(diag(CdMPHI),ncb,ncb);      % variance of CB modal mass uncertainty

Cdk = cov(dkr');                               % covariance matrix for uncertainty in fixed modal stiffness
dvark = reshape(diag(Cdk),nq,nq);              % variance of fixed modal stiffness uncertainty

Cdm = cov(dmr');                               % covariance matrix for uncertainty in fixed modal mass
dvarm = reshape(diag(Cdm),nq,nq);              % variance of fixed modal mass uncertainty

end
```

Table A-4. Function “xrecovermag” recovers uncertainty in frf magnitude and phase.

```
function [dvarmag,dvartheta]=xrecovermag(varC,relS,H,f);
%
%   Created by:      Daniel C. Kammer
%                    Professor
%                    Dept of Engineering Physics
%                    University of Wisconsin
%                    Madison, WI  53706
%                    (608) 262-5724
%
%
%  =================================================================
%
%  This program recovers uncertainty in magnitude and phase for frequency
%  response.  Assumes collocated inputs and outputs.
%
%
%  HISTORY
%  =======
%
%    Created:  Daniel C. Kammer       08-10-11
%
%  =======================================================================
%
%  ns = number of sensors
%  nf = number of frequency data points
%
%  INPUT
%  =====
%  varC       =  diagonal of frf covariance matrices  -  ns^2 x nf
%  relS       =  diagonal of frf relation matrices  -  ns^2 x nf
%  f          =  vector of frequencies in Hz.  -  nf x 1
%  H          =  nominal frf matrix  -  ns x ns x nf
%
%  OUTPUT
%  ======
%
%  dvarmag = variance of frf uncertainty in magnitude
%  dvartheta = variance of frf uncertainty in phase (rad)
%
%  Use:  [dvarmag,dvartheta]=xrecovermag(varC,varS,H,f);
%=========================================================================================================
```

```
%
Date=date
%

nf = size(f,1);                                 % number of frequencies
ns = size(H,1);                                 % number of sensors
%
J = zeros(2,2);
Cri = zeros(2,2);                               % real/imag covariance at kk freq
Cmt = zeros(2,2);                               % magnitude/theta covariance at kk freq

dvarmag = zeros(ns^2,nf);
dvartheta = zeros(ns^2,nf);

for kk = 1:nf                                 % loop over frequencies

    h = H(:,:,kk);
    h = h(:);                                 % vectorize frf matrix at kk freq

    for j = 1:ns^2
        J(1,1) = real(h(j))/abs(h(j));       % construct J matrix
        J(1,2) = imag(h(j))/abs(h(j));
        J(2,1) = -imag(h(j))/(abs(h(j))^2);
        J(2,2) = real(h(j))/(abs(h(j))^2);

        Cri(1,1) = .5*( varC(j,kk) + real(relS(j,kk)) ); % construct Cri matrix
        Cri(2,2) = .5*( varC(j,kk) - real(relS(j,kk)) );
        Cri(1,2) = .5*imag(relS(j,kk));
        Cri(2,1) = Cri(1,2);

        Cmt = J*Cri*J';

        dvarmag(j,kk) = Cmt(1,1);             % varianceof frf magnitude
        dvartheta(j,kk) = Cmt(2,2);           % varianceof frf phase (deg)

    end

end
```